Eigen-values and Eigen-vectors

If $\bar{x} \in V \longrightarrow \underbrace{f}_{\text{ENDOM.}}(\bar{x}) = \underbrace{\lambda}_{\in \mathbb{R}} \bar{x}$

- $\bar{x}$ is an Eigenvector of f for the Eigenvalue λ
- λ is an Eigenvalue of f

Characteristic Equation (C.E.): $\boxed{|F_B - \lambda I| = 0} \longrightarrow \lambda_i$ There are as many Eigenvalues as dim(V).

$S(\lambda) \equiv$ Eigen space for Eigenvalue λ

$$S(\lambda) = \{ \forall \bar{x} \in V / f(\bar{x}) = \lambda \bar{x} \}$$

$$\boxed{S(\lambda) = Ker(f - \lambda i)}$$

$$1 \leqslant \dim(S(\lambda)) \leqslant MO(\lambda)$$

$MO(\lambda) \equiv$ Multiplicity Order of Eigenvalue λ

$$MO(\lambda) = Max(\dim(S(\lambda)))$$

Eigenvalues 1

Given an Endomorph. in $\mathbb{R}^3$, $f(\bar{x}) = (x^1+x^2, x^1+x^2, x^3)$, obtain each and every eigenspace for f with its base and dimension.

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$$\begin{cases} \bar{e}_1 = (1,0,0)_B \\ \bar{e}_2 = (0,1,0)_B \\ \bar{e}_3 = (0,0,1)_B \end{cases}$$

$$F_B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$f(\bar{e}_1)\ f(\bar{e}_2)\ f(\bar{e}_3)$

C.E.:

$$|F_B - \lambda I| = 0 \longrightarrow \left| \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\longrightarrow \underbrace{\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0}_{|F_B - \lambda I| = 0}$$

$$(1-\lambda)\underbrace{\left[(1-\lambda)^2 - 1\right]}_{\lambda^2 - 2\lambda + \cancel{1} \cancel{-1}} = 0 \rightarrow (1-\lambda)(\lambda^2 - 2\lambda) = 0$$

$$(\lambda-2)(1-\lambda)\lambda = 0 \rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = 2 \end{cases}$$

$MO(0) = 1 \rightarrow 1 \leq \dim(S(0)) \leq \cancel{MO(0)}\ 1 \longrightarrow \dim(S(0)) = 1$

$MO(1) = 1 \rightarrow 1 \leq \dim(S(1)) \leq \cancel{MO(1)}\ 1 \longrightarrow \dim(S(1)) = 1$

$MO(2) = 1 \rightarrow 1 \leq \dim(S(2)) \leq \cancel{MO(2)}\ 1 \longrightarrow \dim(S(2)) = 1$

$$S(0) = Ker(f - 0i) = Ker(f)$$

$$\boxed{S(\lambda) = Ker(f - \lambda i)}$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{F_B} \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{0}}$$

$$x^1 + x^2 = 0$$
$$\cancel{x^1 + x^2 = 0}$$
$$x^3 = 0$$

$$\left\{ \begin{array}{l} x^1 = \gamma \\ x^2 = -\gamma \\ x^3 = 0 \end{array} \right. \quad \forall \gamma \in \mathbb{R}$$

$$S(0) = Ker(f) = \mathcal{L}\{\underbrace{(1, -1, 0)}_{\bar{u}_1}\}$$

$$S(1) = Ker(f - i)$$

$$\underbrace{\begin{pmatrix} 1-1 & 1 & 0 \\ 1 & 1-1 & 0 \\ 0 & 0 & 1-1 \end{pmatrix}}_{F-I} \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{0}} \longrightarrow \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{F-I} \underbrace{\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{0}} \longrightarrow \left\{ \begin{array}{l} x^2 = 0 \\ x^1 = 0 \\ \cancel{0 = 0} \end{array} \right.$$

$$\left\{ \begin{array}{l} x^1 = 0 \\ x^2 = 0 \\ x^3 = \alpha \end{array} \right. \quad \forall \alpha \in \mathbb{R}$$

$$S(1) = \mathcal{L}\{\underbrace{(0, 0, 1)}_{\bar{u}_2}\} \quad B_{S(1)} = \{(0, 0, 1)\}$$

$$S(2) = Ker(f - 2i)$$

$F - 2I$ $\quad x$

$$S(2) = \mathcal{L}\{(1, 1, 0)\}$$

$$\begin{cases} \bar{u}_1 = (1,-1,0)_B \in S(0) \\ \bar{u}_2 = (0,0,1)_B \in S(1) \\ \bar{u}_3 = (1,1,0)_B \in S(2) \end{cases}$$

$$\begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -2 \neq 0 \qquad B' = \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$$

They are L.I.

$$\begin{cases} \bar{u}_1 : (1,0,0)_{B'} \\ \bar{u}_2 : (0,1,0)_{B'} \\ \bar{u}_3 : (0,0,1)_{B'} \end{cases}$$

$$f(\bar{u}_1) = 0 \cdot \bar{u}_1 = \bar{0}$$

$$f(\bar{u}_2) = 1 \cdot \bar{u}_2 = \bar{u}_2$$

$$f(\bar{u}_3) = 2 \cdot \bar{u}_3 = 2\bar{u}_3$$

$$F_{B'} = \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ f(\bar{u}_1) \; f(\bar{u}_2) \; f(\bar{u}_3) \end{array}$$

in B'

B' is a base of Eigen Vectors

$F_{B'}$ is a DIAGONAL matrix with Eigenvalues in its MAIN DIAGONAL

$$\begin{tikzcd} B \arrow[r, bend left, "C"] & B' \arrow[l, bend left, "C^{-1}"] \end{tikzcd}$$

$$C = \begin{array}{c} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \bar{u}_1 \; \bar{u}_2 \; \bar{u}_3 \end{array}$$

in B

$$C^{-1} = \frac{Adj(C)^t}{|C|}$$

$$\boxed{F_{B'} = C^{-1} F_B \, C}$$

$\bar{u}_3$

in B'

When is an ENDOMORPHISM's matrix able to be diagonalized?

When I can create a base of EIGENVECTORS.

If $f: V \longrightarrow V$ is an ENDOMORPHISM and B' is a BASE of EIGENVECTORS

$$\boxed{\exists B' \Longleftrightarrow OM(\lambda_i) = \dim(S(\lambda_i)) \quad \forall i}$$

Eigenvalues 2

Is the endomorphism $f(\bar{x}) = (x^1+x^2+x^3,\ x^1+x^2+x^3,\ x^1+x^2+x^3)$ diagonizable ?

$1 \leq \dim(S(\lambda)) \leq MO(\lambda)$

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$$F_B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$f(\bar{e}_1)\ \ f(\bar{e}_2)\ \ f(\bar{e}_3)$

$$\left.\begin{array}{l} f(\bar{e}_1) - f(\bar{e}_2) = \bar{0} \longrightarrow f(\bar{e}_1 - \bar{e}_2) = \bar{0} \longrightarrow \bar{e}_1 - \bar{e}_2 \in Ker(f) = S(0) \\ f(\bar{e}_2) - f(\bar{e}_3) = \bar{0} \longrightarrow f(\bar{e}_2 - \bar{e}_3) = \bar{0} \longrightarrow \bar{e}_2 - \bar{e}_3 \in Ker(f) = S(0) \end{array}\right\}$$

$\dim(S(0)) \geq 2$

so $MO(0) \geq 2$

$$f(\bar{e}_1) + f(\bar{e}_2) + f(\bar{e}_3) = 3\bar{e}_1 + 3\bar{e}_2 + 3\bar{e}_3 \longrightarrow f(\bar{e}_1 + \bar{e}_2 + \bar{e}_3) = 3(\bar{e}_1 + \bar{e}_2 + \bar{e}_3)$$

$\bar{e}_1 + \bar{e}_2 + \bar{e}_3 \in S(3)$ $\quad \dim(S(3)) \geq 1$

so $MO(3) \geq 1$

Since the M.O. of each Eigenvalue is the same as the dimension of the Eigenspaces -> It is DIAGONALIZABLE

Or we solve it in the traditional way:

$$|F_B - \lambda I| = 0 \longrightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ \dots \end{vmatrix} = 0 \longrightarrow [\dots] \longrightarrow (\lambda - 3)\lambda^2 = 0$$